

The Witt Ring of a Smooth Projective Curve over a Finite Field

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Abstract

In this paper we calculate the Witt ring $W(C)$ of a smooth geometrically connected projective curve C over a finite field with characteristic other than 2. We view $W(C)$ as a subring of $W(k(C))$ where $k(C)$ is the function field of C . The calculation is then completed using classical results for bilinear spaces over fields.

Let k be a finite field of characteristic $q \neq 2$. We will show that the Brauer group of a curve C over k vanishes. The vanishing of the Witt invariant then allows us to represent any symmetric space in $W(C)$ by a form of rank one or two and allows us to write out the multiplication and addition table for $W(C)$ and then recognize it as a quotient of $W(k)[_2Pic(C)]$.

For bilinear spaces over fields, we adopt the notation of (Lam, 2005), in which $\langle t \rangle$ denotes the rank 1 space with fixed generator e and whose form takes (e, e) to $t \in k$ and $\langle t_1, \dots, t_n \rangle$ is the orthogonal sum $\langle t_1 \rangle \perp \dots \perp \langle t_n \rangle$.

The Witt ring $W(k)$ of k is a four element ring. It consists of 0, two rank one represented elements $\langle 1 \rangle$ (the multiplicative identity) and $\langle s \rangle$, and a nontrivial even rank element.

The nontrivial even rank element is $\langle 1, s \rangle$ when $q = 1(\text{mod } 4)$ and $\langle 1, 1 \rangle$ when $q = 3(\text{mod } 4)$.

$W(k)$ has a few properties which are useful for calculational purposes.

- $\langle 1, 1 \rangle = \langle s, s \rangle$ for all nondyadic finite fields.
- $\langle 1, 1, 1 \rangle = \langle 1 \rangle = \langle -1 \rangle$ when $q = 1(\text{mod } 4)$.
- $\langle 1, 1, 1 \rangle = \langle s \rangle = \langle -1 \rangle$ when $q = 3(\text{mod } 4)$.
- $\langle 1, 1, 1, 1 \rangle = 0$ for all nondyadic finite fields.

Let C be a smooth geometrically connected projective curve over k with generic point η . Assume also that C contains a k -rational point.

The natural map $W(C) \rightarrow W(k(C)) : E \rightarrow E_\eta$ is injective (Balmer and Walter, 2002) and embeds the Witt ring of C into the Witt ring of its function field. This suggests the following notation for spaces represented by orthogonal sums of rank 1 spaces.

$\langle \mathcal{L}_\xi \rangle$ will denote the Witt class represented by a form which maps to $\langle \xi \rangle \in W(k(C))$ whose underlying vector bundle is the line bundle \mathcal{L} . $\langle \mathcal{L}_{1,\xi_1}, \dots, \mathcal{L}_{n,\xi_n} \rangle$ will denote the orthogonal sum $\langle \mathcal{L}_{1,\xi_1} \rangle \perp \dots \perp \langle \mathcal{L}_{n,\xi_n} \rangle$.

Due to the rational point, the ring map $W(k) \hookrightarrow W(C)$ induced by the structure map $C \rightarrow \text{Spec}(k)$ is also injective. This map identifies $\langle 1 \rangle$ and $\langle s \rangle$ with the Witt classes represented by forms on the structure sheaf.

The following two propositions show that there are $2n$ distinct Witt classes represented by rank 1 elements where n is the cardinality of the order 2 Picard group ${}_2\text{Pic}(C)$.

Proposition 1. *Given $\mathcal{L}, \mathcal{M} \in {}_2\text{Pic}(C)$, if $\langle \mathcal{L}_\xi \rangle = \langle \mathcal{M}_\zeta \rangle \in W(C)$ then $\mathcal{L} = \mathcal{M} \in {}_2\text{Pic}(C)$ and $\langle \xi \rangle = \langle \zeta \rangle \in W(k(C))$.*

Proof. That $\langle \xi \rangle = \langle \zeta \rangle \in W(k(C))$ is trivial.

Let $\mathcal{L}, \mathcal{M} \in {}_2\text{Pic}(C)$ and $\langle \mathcal{L}_\xi \rangle = \langle \mathcal{M}_\zeta \rangle \in W(C)$. Then $\langle \mathcal{L}_\xi \rangle \perp M = \langle \mathcal{M}_\zeta \rangle \perp M' \in \text{Bil}(C)^1$ where M and M' are metabolic spaces of equal rank $2m$.

Taking determinants (Knebusch, 1977) on both sides, we see that $\det(\langle \mathcal{L}_\xi \rangle) \det(M) = \det(\langle \mathcal{M}_\zeta \rangle) \det(M')$, $\langle \mathcal{L}_\xi \rangle (\langle -1 \rangle^m) = \langle \mathcal{M}_\zeta \rangle (\langle -1 \rangle^m)$, and $\langle \mathcal{L}_\xi \rangle = \langle \mathcal{M}_\zeta \rangle \in \text{Bil}(C)$.

Proposition 2. *Given $\mathcal{L} \in {}_2\text{Pic}(C)$, there are two Witt classes represented by forms whose underlying space is \mathcal{L} .*

Proof. First, we note that every order 2 line bundle \mathcal{L} is equipped with an isomorphism $\mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_C$, which may be used to define a nondegenerate symmetric bilinear form on \mathcal{L} .

Two rank 1 forms (\mathcal{L}, φ) and (\mathcal{L}, ψ) may differ at most by a global endomorphism of \mathcal{L} as shown in the following diagram:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\varphi} & \mathcal{L}^\vee \\ \varphi\psi^{-1} \downarrow & & \parallel \\ \mathcal{L} & \xrightarrow{\psi} & \mathcal{L}^\vee \end{array}$$

As $\mathcal{E}nd(\mathcal{L}) \cong \mathcal{O}_C$, the global endomorphisms of \mathcal{L} are precisely the units of k . Thus, φ, ψ differ by multiplication m_ℓ by some unit $\ell \in k$.

¹ $\text{Bil}(C)$ is the semiring consisting of isomorphism classes of bilinear spaces on C

φ and ψ represent the same Witt class precisely when $\varphi = m_\ell \circ \psi \circ m_\ell^\vee$ so that $\varphi = \ell^2 \psi$.

This means that the Witt classes of C associated to \mathcal{L} are in one-to-one correspondence with the square classes of k , which correspond to the rank 1 elements of $W(k)$.

We note that multiplication by $\langle s \rangle \neq \langle 1 \rangle \in W(k)$ exchanges the two classes of $W(C)$ represented by forms on \mathcal{L} .

Since the function field of C is a C_2 field we may apply the following classical result for fields (Lam, 2005) Proposition V.3.25.

Proposition 3. *Suppose every form of dimension 5 over a field F is isotropic. Then two bilinear spaces E, E' are isometric iff $rk(E) = rk(E') \in \mathbb{Z}$, $d_\pm(E) = d_\pm(E') \in W(F)$ and $c(E) = c(E') \in Br(F)$ where $d_\pm(-)$ is the signed discriminant and the Witt invariant $c(-)$ is the class of the Clifford algebra in the Brauer group $Br(-)$.*

The signed discriminant of E is the rank 1 form $(-1)^{\frac{n(n+1)}{2}} \bigwedge^n E$ where n is the rank of E . Note that the signed discriminant of a form $\langle a_1, \dots, a_n \rangle$ is $(-1)^{\frac{n(n+1)}{2}} \langle a_1 \cdots a_n \rangle$.

Details regarding Clifford algebras for forms over fields may be found in (Lam, 2005) Chapter V. A similar definition is used for bilinear spaces on schemes. In the current situation it suffices to know the following:

- $c(E) \in Br(C)$.
- The natural map $Br(C) \rightarrow Br(k(C)) : A \mapsto A_\eta$ is injective.
- $c(E)_\eta = c(E_\eta)$.

The following theorem now shows that every bilinear space over C has trivial Witt invariant over C and $k(C)$.

Theorem 1. $Br(C) = 0$

Proof. We calculate the cohomological Brauer group $H^2(C_{et}, \mathbb{G}_m)$.

There is a spectral sequence $H^p(\bar{G}, H^q(\bar{C}, \mathbb{G}_m)) \Rightarrow H^{p+q}(C_{et}, \mathbb{G}_m)$ (Milne, 1980) III.1.16, 2.20 where \bar{C} is the extension of C to the separable closure k_{sep} of the base field and $\bar{G} = Gal(k_{sep}/k)$. Since C is a curve over a finite field, \bar{C} is also the extension to the algebraic closure.

This is a first quadrant spectral sequence with E^2 terms as follows:

$E_{0,2}^2 = H^0(\bar{G}, H^2(\bar{C}, \mathbb{G}_m))$ and $H^2(\bar{C}, \mathbb{G}_m) = Br(\bar{C}) = 0$ due to the fact that $k(\bar{C})$ is a C_1 field (Tsen's theorem). Thus, $E_{0,2}^2 = 0$.

$$E_{2,0}^2 = H^2(\bar{G}, H^0(\bar{C}, \mathbb{G}_m)) = H^2(\bar{G}, k_{sep}^\times) = Br(k) = 0$$

$$E_{1,1}^2 = H^1(\bar{G}, H^1(\bar{C}, \mathbb{G}_m)) = H^1(\bar{G}, Pic(\bar{C})). \text{ There is a short exact sequence}$$

$$0 \longrightarrow Pic^0(\bar{C}) \longrightarrow Pic(\bar{C}) \xrightarrow{deg} \mathbb{Z} \longrightarrow 0$$

Furthermore, $H^1(\bar{G}, Pic^0(\bar{C})) = 0$ (Lang's Theorem) and

$$H^1(\bar{G}, \mathbb{Z}) = Hom_{cont}(\bar{G}, \mathbb{Z}) = 0 \text{ so that } E_{1,1}^2 = H^1(\bar{G}, Pic(\bar{C})) = 0.$$

This shows that the cohomological Brauer group, hence the Brauer group, is trivial.

Theorem 1 gives relations $\langle \mathcal{L}_\xi, \mathcal{M}_\zeta \rangle = \langle 1, \mathcal{L} \mathcal{M}_{\xi\zeta} \rangle$ in $W(C)$ as well as allowing us to show the following.

Proposition 4. *Every class $E \in W(C)$ has a representative of the form $\langle \mathcal{L}_\xi \rangle$ or*

$$\langle 1, -\mathcal{L}_\xi \rangle \text{ where } \langle \mathcal{L}_\xi \rangle = d_\pm(E).$$

Proof. Consider the image E_η of E in the function field.

E_η has an anisotropic representative V of rank less than five. We will show that this representative must, in fact, have rank less than three.

We first note that the Witt invariant $c(V)$ is trivial.

The tertiary case now follows from (Lam, 2005) Proposition V.3.22.

It remains to show that a rank four form cannot be anisotropic. We consider a diagonalisation $\langle a_1, a_2, a_3, a_4 \rangle$ of V . Since $c(V)$ is trivial, Proposition 3 shows that V is isometric to $\langle 1, 1, 1, a_1 a_2 a_3 a_4 \rangle = \langle -1, a_1 a_2 a_3 a_4 \rangle \in W(k(C))$.

Calculations such as $\langle 1, -\mathcal{L}_\xi \rangle \perp \langle \mathcal{M}_\zeta \rangle = \langle 1, 1, -\mathcal{L} \mathcal{M}_{\xi\zeta} \rangle = \langle \mathcal{L} \mathcal{M}_{\xi\zeta} \rangle$ now give us the following tables of arithmetic for $W(C)$.

Multiplication:

	$\langle \mathcal{L}_\xi \rangle$	$\langle 1, -\mathcal{L}'_{\xi'} \rangle$
$\langle \mathcal{M}_\zeta \rangle$	$\langle \mathcal{L} \mathcal{M}_{\xi\zeta} \rangle$	$\langle 1, -\mathcal{L}'_{\xi'} \rangle$
$\langle 1, -\mathcal{M}'_{\zeta'} \rangle$	$\langle 1, -\mathcal{M}'_{\zeta'} \rangle$	0

Note, in particular, that $\langle \mathcal{L}_\xi \rangle \langle \mathcal{L}_\xi \rangle = \langle 1 \rangle$ and

$$\langle \mathcal{L}_\xi \rangle \langle \mathcal{L}_{s\xi} \rangle = \langle s \rangle$$

Addition:

	$\langle \mathcal{L}_\xi \rangle$	$\langle 1, -\mathcal{L}'_{\xi'} \rangle$
$\langle \mathcal{M}_\zeta \rangle$	$\langle 1, \mathcal{L} \mathcal{M}_{\xi\zeta} \rangle$	$\langle \mathcal{M} \mathcal{L}'_{\xi'\zeta} \rangle$
$\langle 1, -\mathcal{M}'_{\zeta'} \rangle$	$\langle \mathcal{M}' \mathcal{L}'_{\xi'\zeta} \rangle$	$\langle 1, -\mathcal{M}' \mathcal{L}'_{\xi'\zeta} \rangle$

Note that $\langle \mathcal{L}_\xi \rangle \perp \langle \mathcal{L}_\xi \rangle = \langle 1, 1 \rangle$ and $\langle \mathcal{L}_\xi \rangle \perp \langle \mathcal{L}_{s\xi} \rangle = \langle 1, s \rangle$.

In fact, $W(C)$ can be expressed quite nicely as a quotient of the group ring $W(k)[_2 \text{Pic}(C)]$.

Theorem 2. $W(C) \cong W(k)[{}_2\text{Pic}(C)]/\mathcal{R}$ where \mathcal{R} is generated by the relations of the form

$$\langle 1 \rangle - \langle u \rangle \mathcal{L} - \langle v \rangle \mathcal{M} + \langle uv \rangle \mathcal{L}\mathcal{M}$$

with $\langle u \rangle, \langle v \rangle \in W(k)$ and $\mathcal{L}, \mathcal{M} \in {}_2\text{Pic}(C)$.

Proof. Fix a form $\langle \mathcal{L}_\xi \rangle$ on each $\mathcal{L} \in {}_2\text{Pic}(C)$.

Define a map $W(k)[{}_2\text{Pic}(C)]/\mathcal{R} \rightarrow W(C)$ by sending $\langle s \rangle \mathcal{L}$ to $\langle \mathcal{L}_{s\xi} \rangle$ and extending by linearity.

This map is clearly a well defined surjection of commutative rings.

To show injectivity let $f \in W(k)[{}_2\text{Pic}(C)]/\mathcal{R}$ map to a form

$E = \langle \mathcal{L}_{1,u_1\xi_1}, \dots, \mathcal{L}_{n,u_n\xi_n} \rangle = 0 \in W(C)$. Using the relations $\langle 1, 1 \rangle = \langle -1, -1 \rangle$ and $\langle \mathcal{L}_{u\xi}, \mathcal{M}_{v\xi} \rangle = \langle 1, \mathcal{L}\mathcal{M}_{uv\xi\xi} \rangle$ we may rewrite E as $\langle \pm 1, \mathcal{L}_1 \cdots \mathcal{L}_{n,u_1 \cdots u_n \xi_1 \cdots \xi_n} \rangle$. Using the corresponding relations $\langle 1 \rangle + \langle 1 \rangle + \langle 1 \rangle + \langle 1 \rangle$ and $\langle 1 \rangle - \langle u \rangle \mathcal{L} - \langle v \rangle \mathcal{M} + \langle uv \rangle \mathcal{L}\mathcal{M}$ in $W(k)[{}_2\text{Pic}(C)]/\mathcal{R}$, we may write f as $\langle \pm 1 \rangle + \langle u_1 \cdots u_n \rangle \mathcal{L}_1 \cdots \mathcal{L}_n$. In the $+1$ case, E and f are trivial precisely when $\langle u_1 \cdots u_n \mathcal{L}_1 \cdots \mathcal{L}_n \rangle = \langle 1 \rangle$. In the -1 case, both E and f are trivial precisely when $\langle u_1 \cdots u_n \mathcal{L}_1 \cdots \mathcal{L}_n \rangle = \langle -1 \rangle$.

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